Section 4.2

Arclength:

Q: Suppose you are driving and your motion is described by a path $\vec{c}(t) = (x(t), y(t), z(t))$

Then what is the length of the path you trace between time t_0 & time t_1?

A: $$L(\vec{c}) = \int_{t_0}^{t_1} \underbrace{\|\vec{c}'(t)\|}_{\text{speed}}\, dt$$

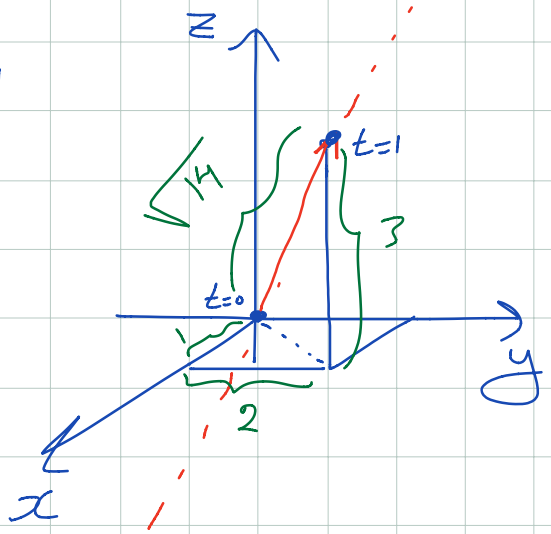
(Essentially, we are saying that dist = speed x time)

Let's check on a couple of easy examples

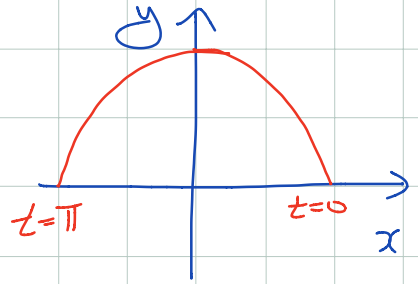
$$\vec{c}(t) = (0,0,0) + (1,2,3)\,t \quad , \quad t_0 = 0,\ t_1 = 1$$

$$\|\vec{c}'(t)\| = \|(1,2,3)\|$$
$$= \sqrt{1^2+2^2+3^2}$$
$$= \sqrt{14}$$

$$L(\vec{c}) = \int_0^1 \sqrt{14}\, dt$$
$$= \sqrt{14}$$



How about: $\vec{c}(t) = (\cos t, \sin t)$, $t_0 = 0$, $t_1 = \pi$



$$L(\vec{c}) = \int_0^{\pi} \|c'(t)\|\, dt = \int_0^{\pi} \|(-\sin t, \cos t)\|\, dt$$

$$= \int_0^{\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= \int_0^{\pi} dt = t\Big|_0^{\pi} = \pi$$

↑ half the perimeter of a circle of radius 1

Let's do one where we don't know the answer

$$\vec{c}(t) = (\cos t, \sin t, t), \quad t_0 = 0, \quad t_1 = 2\pi$$

$L(\vec{c}) = ?$

$$\vec{c}'(t) = (-\sin t, \cos t, 1)$$

$$\Rightarrow \|c'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$\Rightarrow L(\vec{c}) = \int_0^{2\pi} \|\vec{c}'(t)\|\, dt = \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2}\, 2\pi$$

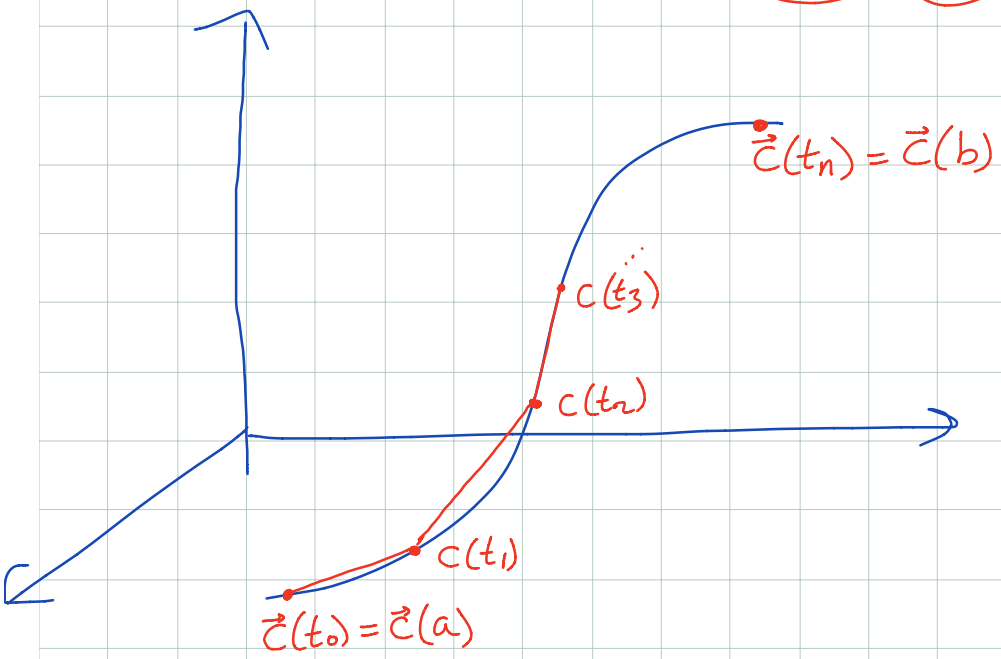
Example: $\vec{c}(t) = (\cos(t^2), \sin(t^2))$, $t_0 = 0$, $t_1 = \pi$

$$\vec{c}\,'(t) = (-2t\sin t, 2t\cos t)$$

$$\|\vec{c}\,'(t)\| = \sqrt{4t^2} = 2t$$

$$L(\vec{c}) = \int_0^{\pi} 2t\,dt = t^2\Big|_0^{\pi} = \pi^2$$

Justification of the arclength formula



Suppose, we wanted the arclength between $\vec{c}(a)$ & $\vec{c}(b)$

Let's call $a = t_0$ & $b = t_N$
and let's take N time points between a & b:

$$t_0 < t_1 < t_2 \cdots < t_N$$

and let's connect $\vec{c}(t_0)$ to $\vec{c}(t_1)$ by a line segment
$\vec{c}(t_1)$ to $\vec{c}(t_2)$ by a line segm—

$\vec{c}(t_i)$ to $\vec{c}(t_{i+1})$ by a line seg.

The length of the line segment:

between $\vec{c}(t_i) = (x(t_i), y(t_i), z(t_i))$
& $\vec{c}(t_{i+1}) = (x(t_{i+1}), y(t_{i+1}), z(t_{i+1}))$

is

$$\|\vec{c}(t_{i+1}) - \vec{c}(t_i)\| = \sqrt{(x(t_{i+1}) - x(t_i))^2 + (y(t_{i+1}) - y(t_i))^2 + (z(t_{i+1}) - z(t_i))^2}$$

$$= \sqrt{(x'(\alpha_i)(t_{i+1} - t_i))^2 + (y'(\beta_i)(t_{i+1} - t_i))^2 + (z'(\gamma_i)(t_{i+1} - t_i))^2}$$

for some $\alpha_i, \beta_i, \gamma_i \in (t_i, t_{i+1})$ by the mean value theorem

So $\|\vec{c}(t_{i+1}) - \vec{c}(t_i)\| = \sqrt{x'(\alpha_i)^2 + y'(\beta_i)^2 + z'(\gamma_i)^2}\ (t_{i+1} - t_i)$

$\Rightarrow$ total length of all line segments is

$$S_n = \sum_{i=0}^{n} \sqrt{x'(\alpha_i)^2 + y'(\beta_i)^2 + z'(\gamma_i)^2}\ (t_{i+1} - t_i)$$

Taking more and more line segments we approximate $L(\vec{c})$ more closely, so that

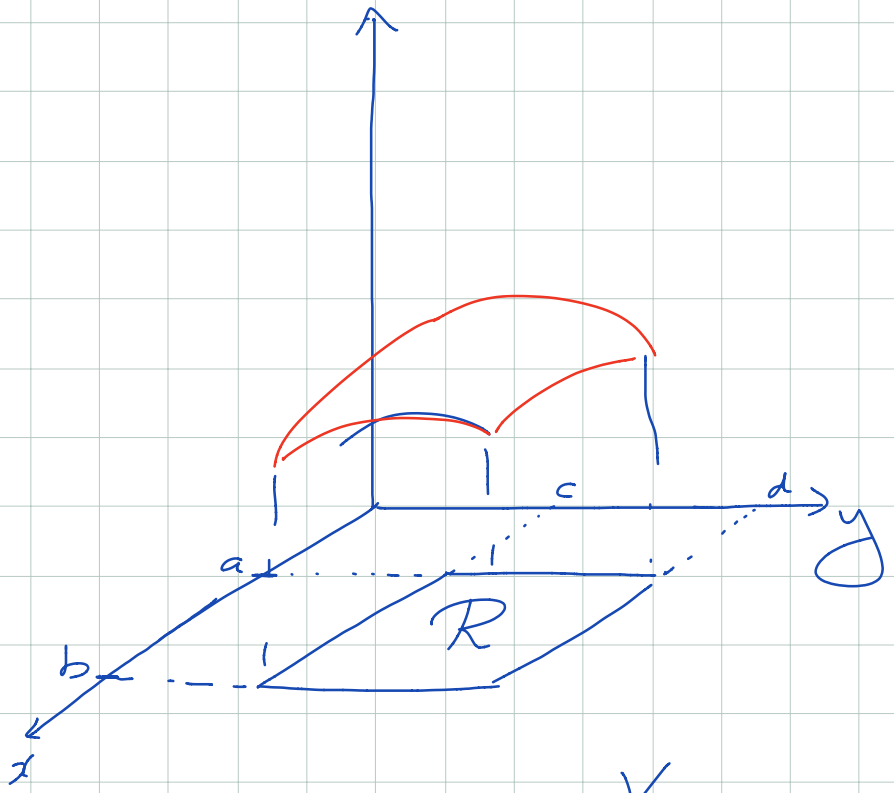
$$L(\vec{c}) = \lim_{n\to\infty} S_n$$

$$= \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

$$= \int_a^b \|\vec{c}'(t)\|\, dt.$$

Ch 5 Double & Triple Integrals

Sec 5.1



Want to find the volume under the graph of the functions

$$z = f(x,y) \quad (f(x,y) \geq 0)$$

and above the rectangle $[a,b] \times [c,d]$

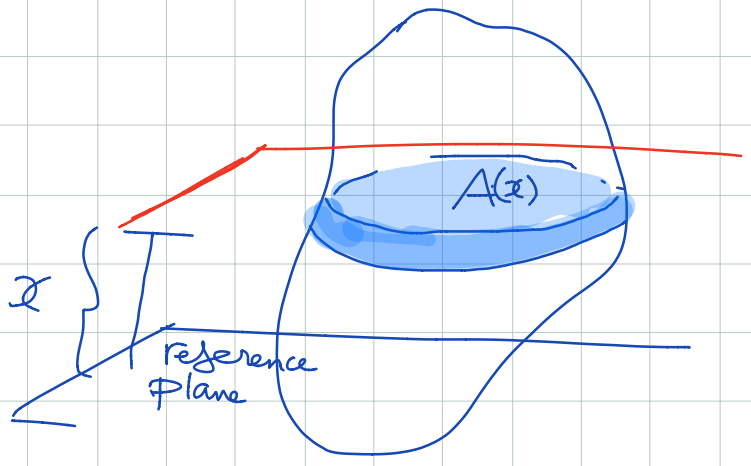
Double Integral

The Volume above the region R and under the graph of $f(x,y)$ $(f(x,y) \geq 0)$

is called the double integral of f over R and we denote it by

$$\iint_R f(x,y)\,dA \quad \text{or} \quad \iint_R f(x,y)\,dx\,dy$$

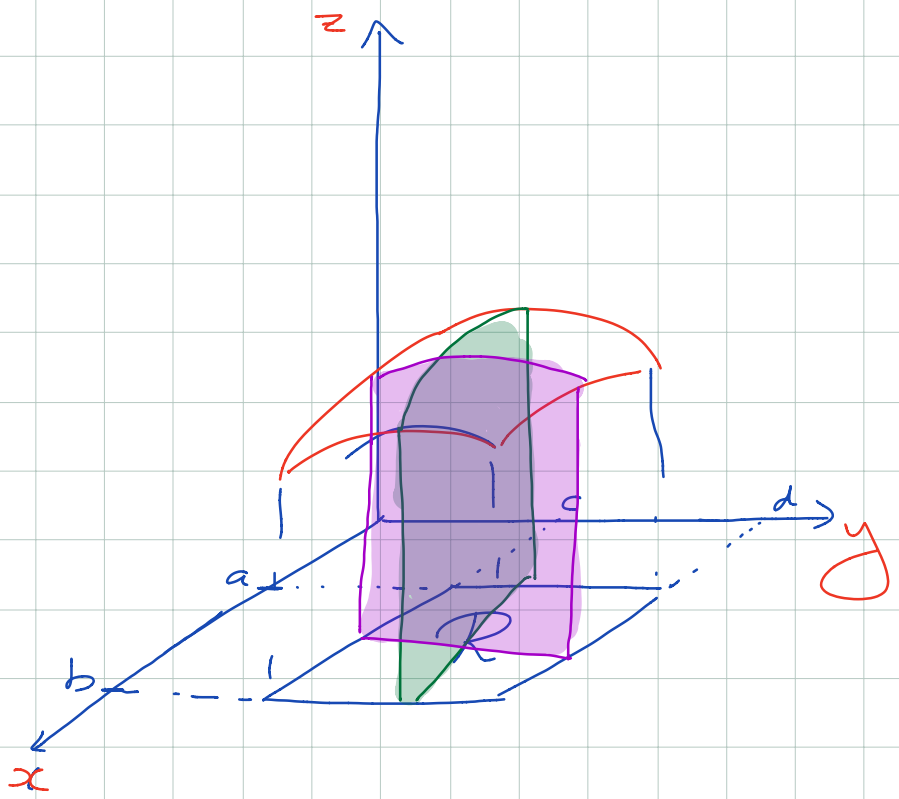
The Slice Method for computing Volumes



Idea: The volume of the object is the sum of the volumes of thin slices of it

$$\Rightarrow \text{Volume} = \int_a^b A(x)\,dx$$

Let's now use the slice method to get something nicer:



Slicing parallel to the yz-plane

$$V \overset{\text{slice principle}}{=} \int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

Slicing parallel to the xz-plane

$$V = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

Iterated integrals

Examples:

Evaluate the integral

$$\iint_R x^2+y^2\,dA \quad \text{where } R=[-1,1]\times[0,1]$$

Sol'n

Using iterated integrals:

$$\iint_R (x^2+y^2)\,dA = \int_{-1}^{1}\left[\int_0^1 x^2+y^2\,dy\right]dx$$

$$= \int_{-1}^{1}\left(x^2y+\frac{y^3}{3}\right)\Big|_{y=0}^{y=1}dx$$

$$= \int_{-1}^{1}\left(x^2+\frac{1}{3}\right)dx$$

$$= x^3/3 + x/3\,\Big|_{-1}^{1} = \frac{2}{3}-\left(-\frac{2}{3}\right)=\frac{4}{3}$$

Let's do it again in the other order:

$$\iint_R x^2+y^2\,dA = \int_0^1\left[\int_{-1}^{1} x^2+y^2\,dx\right]dy$$

$$= \int_0^1 \frac{x^3}{3} + xy^2 \Big|_{x=-1}^{x=1} dy$$

$$= \int_0^1 \frac{2}{3} + 2y^2 \, dy = \frac{2}{3}y + \frac{2}{3}y^3 \Big|_0^1$$

$$= 4/3$$

We just found the volume under the paraboloid $z = x^2 + y^2$ and above the rectangle $[-1,1] \times [0,1]$!

Example

Find $\int_0^1 \int_0^1 xy e^{x+y} \, dx \, dy$

Sol'n

$$\int_0^1 \int_0^1 xy e^{x+y} \, dx \, dy = \int_0^1 \int_0^1 (xe^x)(ye^y) \, dx \, dy$$

But $\int_0^1 xe^x dx = xe^x - e^x$

So $\int_0^1\int_0^1 xe^x ye^y\,dx\,dy = \int_0^1 (xe^x - e^x)ye^y\Big|_{x=0}^{1} dy$

$= \int_0^1 (1)ye^y\,dy = (1)(ye^y - e^y)\Big|_0^1$

$= 1$

Example

Evaluate

$$V = \int_0^2\int_{-1}^{0} -xe^x \sin\frac{\pi}{2}y\,dy\,dx$$

Soln

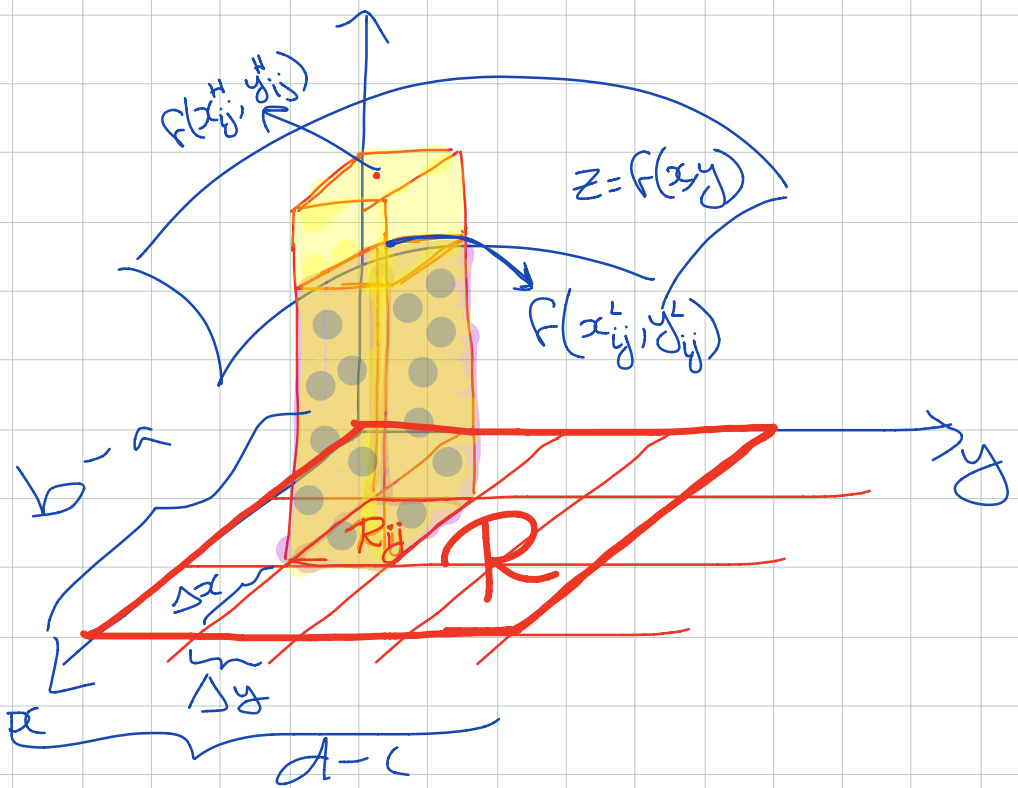
$V = \int_0^2 +\frac{2}{\pi}xe^x \cos\frac{\pi}{2}y\Big|_{y=-1}^{0} dx$

$= \int_0^2 \frac{2}{\pi}xe^x(1-0)\,dx = \frac{2}{\pi}\int_0^2 xe^x\,dx$

$= \frac{2}{\pi}(xe^x - e^x)\Big|_{x=0}^{x=2} = \frac{2}{\pi}(2e^2 - e^2 - 1)$

$= \frac{2}{\pi}(e^2 - 1)$

5.2 Double integrals over a rectangle



The volume above the rectangle R_{ij} and below the graph of $z = f(x,y)$ is

1) Less than the volume of the yellow rectang. box

2) Greater than the volume of the dotted box.

Taking limits as the number of rectangles R_{ij} increases and summing all the volumes:

If

$$\lim_{n\to\infty} \sum_{ij=0}^{n-1} f(x_{ij}^{L}, y_{ij}^{L})\,\Delta x\,\Delta y$$

$$= \lim_{n\to\infty} \sum_{ij=0}^{n-1} f(x_{ij}^{H}, y_{ij}^{H})\,\Delta x\,\Delta y$$

$$= \lim_{n\to\infty} \sum_{i,j=0}^{n-1} f(x_{ij}, y_{ij})\,\Delta x\,\Delta y = S$$

(converges to a limit S)

We say f is <u>integrable over R</u>

The integral is written

$$\iint_R f(x,y)\,dA \quad \text{or} \quad \iint_R f(x,y)\,dx\,dy$$

$$\text{or} \quad \iint_R f\,dA$$

<u>Theorem</u>: A continuous function on a closed rectangle is integrable

Properties: Same as single variable integration

- $\iint_R (f+g)\,dA = \iint_R f\,dA + \iint_R g\,dA$
- $\iint_R cf\,dA = c\iint_R f\,dA$
- $\iint_{R_1 \cup R_2} f\,dA = \iint_{R_1} f\,dA + \iint_{R_2} f\,dA$ when $R_1 \cap R_2 = \phi$

Theorem: (Fubini) Let $R = [a,b]\times[c,d]$ and let f be continuous on R.

then $\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$

↑ you can change the order of integration!

Example: Compute $\iint_R (x^2+y)\,dA$ $R=[0,1]\times[0,1]$

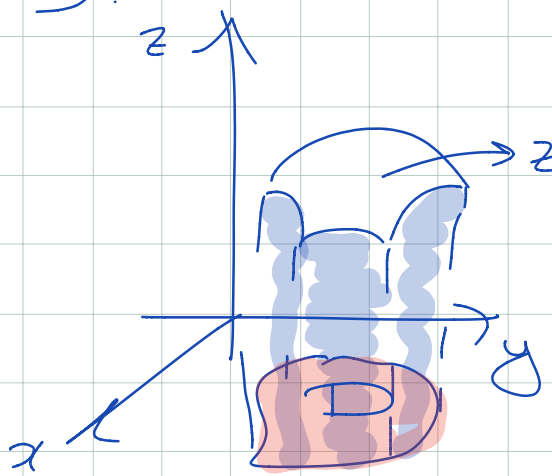
Soln: $\iint_R (x^2+y)\,dA = \int_0^1\int_0^1 (x^2+y)\,dx\,dy$

$$= \int_0^1 \left[\frac{x^3}{3}+xy\right]_{x=0}^{x=1} dy$$

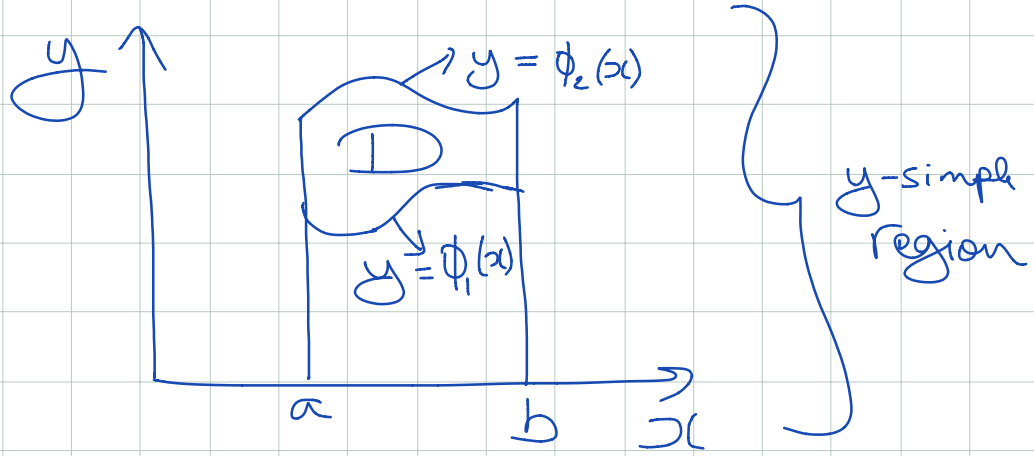
$$= \int_0^1 \frac{1}{3}+y\,dy = \frac{1}{3}+\frac{1}{2} = \frac{5}{6}$$

5.3 Double integrals over more general Regions:

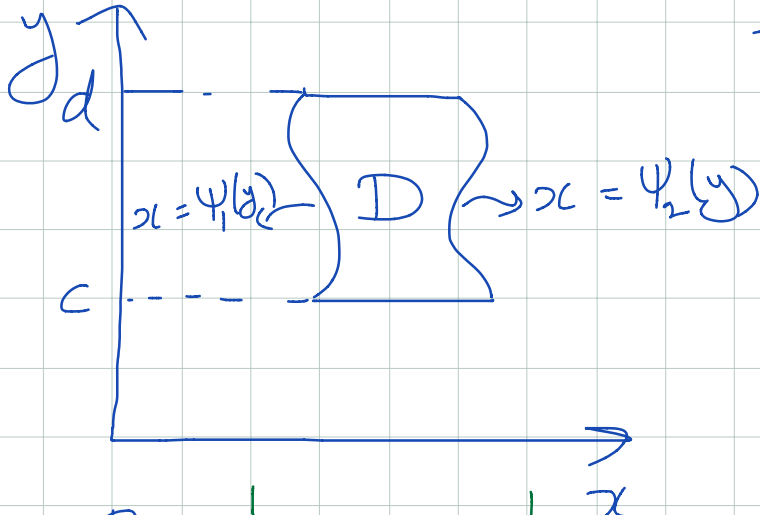
Want the volume "under" the graph of a function and above A general (non-rectangular) region D.



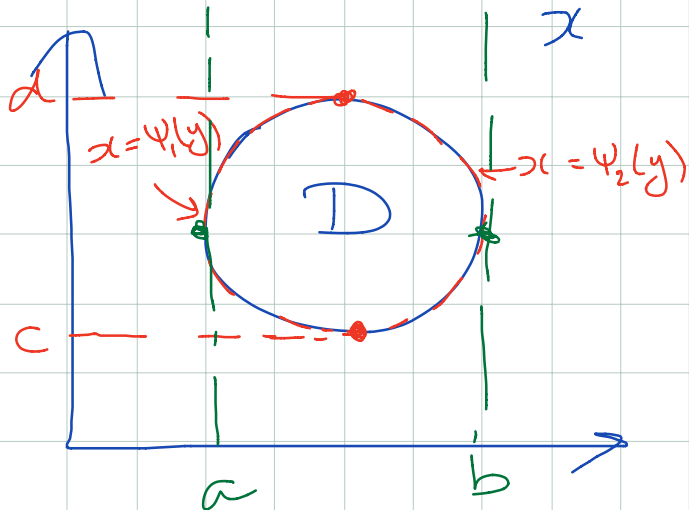
Elementary Regions:



y-simple region



x-simple region



simple region (both x-simple & y-simple)

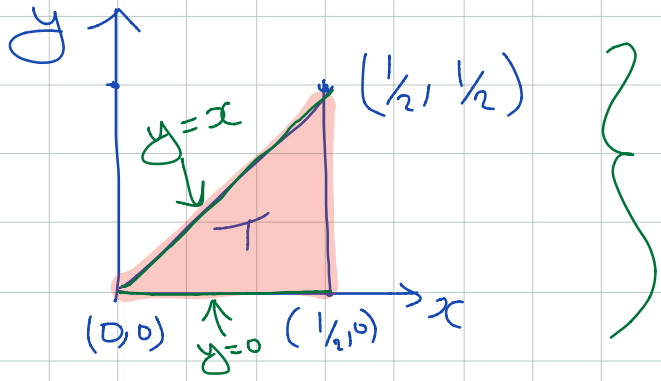
Example:

Find $\iint_T (y + x^2)\, dA$

where T is the triangle with vertices

$(0,0), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2})$.

Soln **Step 1**: Sketch the region



Step 2

$$\iint_T f(x,y)\, dA = \int_0^{\frac{1}{2}} \int_0^x y + x^2 \, dy\, dx$$

$$= \int_0^{1/2} y^2/2 + yx^2 \Big|_{y=0}^{y=x} dx$$

$$= \int_0^{1/2} x^2/2 + x^3 \, dx = \frac{x^3}{6} + \frac{x^4}{4} \Big|_0^{1/2}$$

In general: If D is a simple y-region

$$\iint_D f(x,y)\,dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx$$

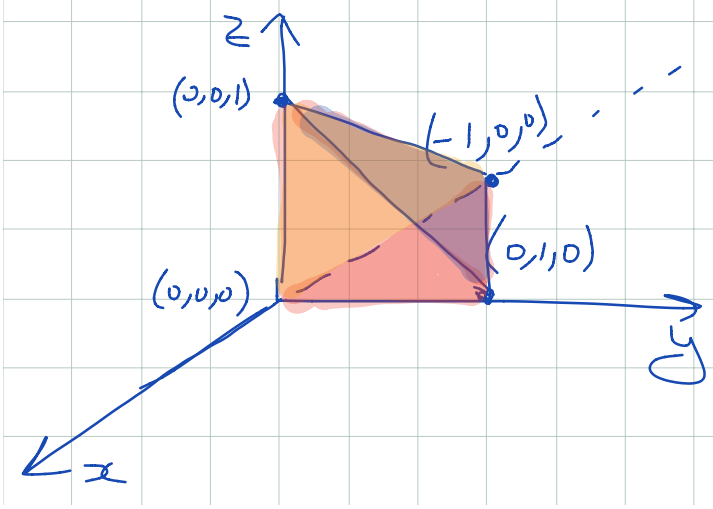
In general: If D is a simple x-region

$$\iint_D f(x,y)\,dA = \int_c^d \int_{\psi_1(x)}^{\psi_2(x)} f(x,y)\,dx\,dy$$

Example:

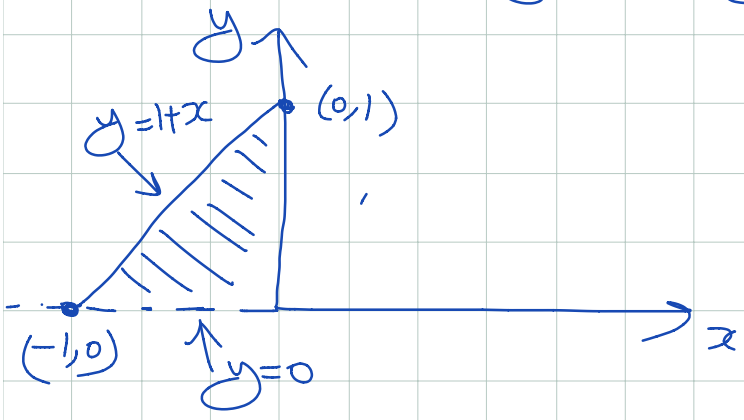
Find the Volume of the tetrahedron bounded by the planes $y=0$, $z=0$, $x=0$ & $-x+y+z=1 \iff z=1+x-y$

Soln: Step 1: Sketch!



So on the xy-plane ($z=0$)

$x=0,\ y=0,\ -x+y=1$



$$\Rightarrow V(\text{tetrahedron}) = \int_{x=-1}^{x=0} \int_{y=0}^{y=1+x} 1+x-y \, dy\, dx$$

$$= \int_{-1}^{0} y + xy - y^2/2 \Big|_{y=0}^{1+x} dx$$

$$= \int_{-1}^{0} \underbrace{(1+x) + (1+x)x - (1+x)^2/2 - 0}_{1+x+x+x^2-\frac{(1+x)^2}{2}} dx$$

$$= \int_{-1}^{0} \frac{1}{2} + x + \frac{x^2}{2}\, dx = \frac{x}{2} + \frac{x^2}{2} + \frac{x^3}{6}\Big|_{-1}^{0}$$

$$= 0 - \left(-\frac{1}{2} + \frac{1}{2} - \frac{1}{6}\right) = \frac{1}{6}$$

$\square$

Sec 5.4:

Sometimes evaluating an iterated integral can be hard so we may need to change the order of integration

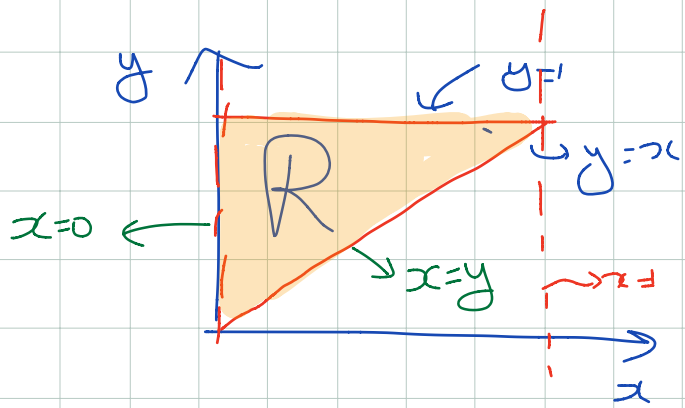
e.g. $\int_0^1 \int_x^1 e^{y^2}\,dy\,dx$

Here,

e^{y^2} doesn't have an antiderivative that we know

So how do we change the order of integration?

We draw



R is both x-simple & y-simple

so we can also define this region by $\underset{\psi_2(y)}{0} \le x \le \underset{\psi_1(y)}{y}$

$0 \le y \le 1$

So $\int_0^1 \int_x^1 e^{y^2}\,dy\,dx = \int_0^1 \int_0^y e^{y^2}\,dx\,dy$

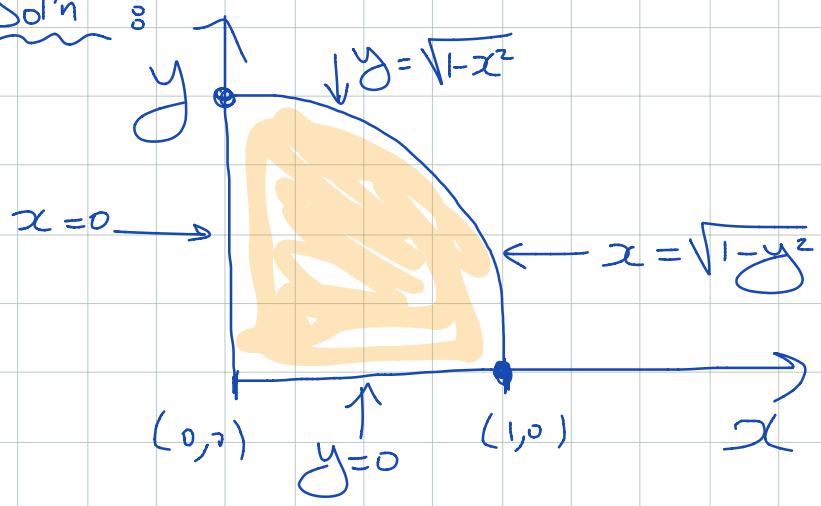
$= \int_0^1 x e^{y^2}\Big|_{x=0}^{x=y} dx\,dy = \int_0^1 y e^{y}\,dy$

$= \frac{1}{2} e^{y^2}\Big|_0^1 = \frac{e-1}{2}$

Example: Evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\, dy\, dx$$

Soln:



The region is both x & y simple, so

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\, dy\, dx = \int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{1-y^2}\, dx\, dy$$

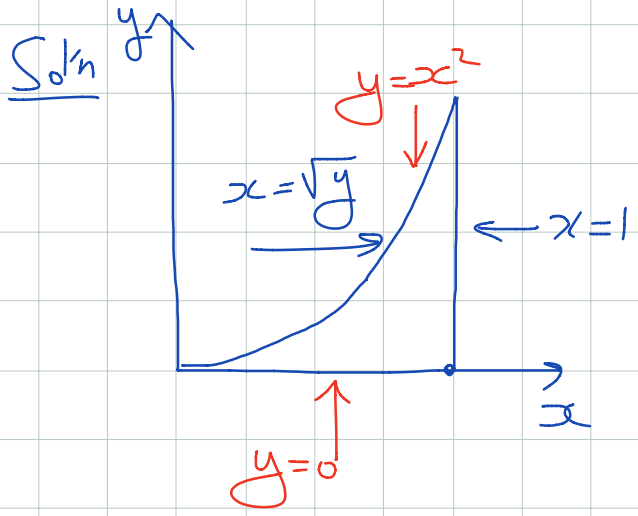
$$= \int_0^1 x\sqrt{1-y^2}\,\Big|_0^{\sqrt{1-y^2}}\, dy$$

$$= \int_0^1 1-y^2\, dy$$

$$= y - y^3/3\,\Big|_0^1 = 2/3 \quad -$$

Example: $\int_0^1 \int_{\sqrt{y}}^1 e^{x^3} dx\, dy$

Soln



$$\int_0^1 \int_{\sqrt{y}}^1 e^{x^3} dx\, dy = \int_0^1 \int_{y=0}^{x^2} e^{x^3} dy\, dx$$

$$= \int_0^1 y e^{x^3} \Big|_{y=0}^{y=x^2} dx$$

$$= \int_{x=0}^{x=1} x^2 e^{x^3} dx, \quad \text{Let } u = x^3, \; du = 3x^2 dx$$

$$= \int_{u=0}^{u=1} \frac{1}{3} e^u du$$

$$= e/3$$

Exercise: do $\int_0^4 \int_{\sqrt{y}}^1 e^{x^3} dx\, dy$

5.5 The triple integral

$$\iiint_R f(x,y,z)\, dV$$

$dV = dA\, dz = dx\,dy\,dz$

R → solid in space

Like Before, we start with integrals over a box (rectangular parallelpiped) $B = [a,b] \times [c,d] \times [p,q]$

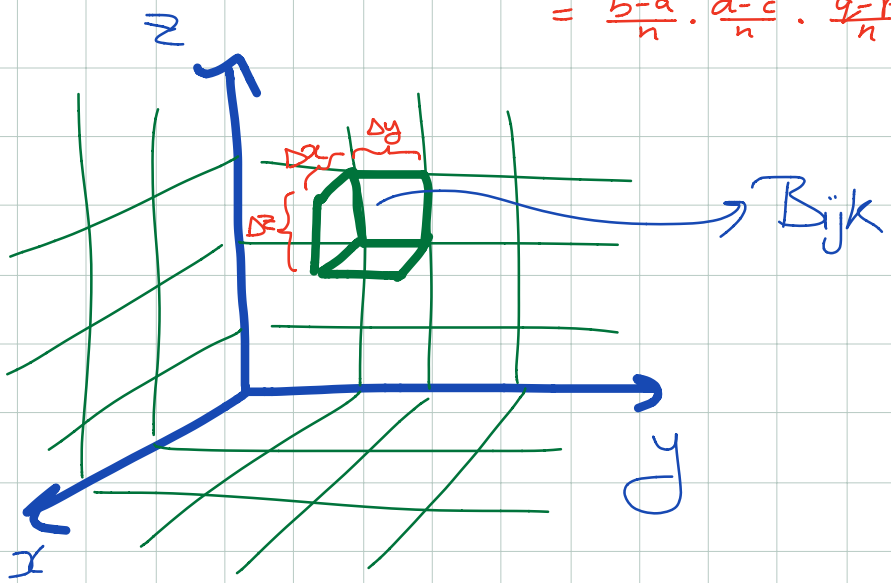
Similar to double and single integrals:

Define $S_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} f(\vec{c}_{ijk})\, \Delta V$

$\vec{c}_{ijk} \in B_{ijk}$, the ijk'th box

$= \Delta x\, \Delta y\, \Delta z$

$= \frac{b-a}{n} \cdot \frac{d-c}{n} \cdot \frac{q-p}{n}$



Def'n: If f is a bounded function of 3 variables on B and if $S = \lim_{n\to\infty} S_n$ exists (and is independent of the choice of $\vec{c}_{ijk} \in B_{ijk}$) then f is integrable and $S =: \iiint_B f(x,y,z)\,dV$ is the triple integral of f over B.

Properties: If f is integrable over B, then the triple integral can be evaluated as an iterated integral

example: Let B be the box given by $0 \le x \le 1$, $0 \le y \le 2$, $-1 \le z \le 0$

evaluate $\iiint_B x^2 + xy + z^2 y\,dV$

$$= \int_0^1 \int_0^2 \int_{-1}^0 x^2 + xy + z^2 y\,dz\,dy\,dx \quad \text{(iterated integral)}$$

$$= \int_0^1 \int_0^2 x^2 z + xyz + \frac{z^3}{3} y \Big|_{z=-1}^{z=0} dy\,dx$$

$$= \int_0^1 \int_0^2 x^2 + xy + \frac{1}{3} y\,dy\,dx$$

$$= \int_0^1 x^2 y + \frac{xy^2}{2} + \frac{y^2}{6} \Big|_{y=0}^{y=2} dx$$

$$= \int_0^1 2x^2 + 2x + 2/3\,dx = 2/3 + 1 + 2/3 = 7/3$$

Exercise: Evaluate the triple integral in a different order.

Next Step: Elementary Regions:

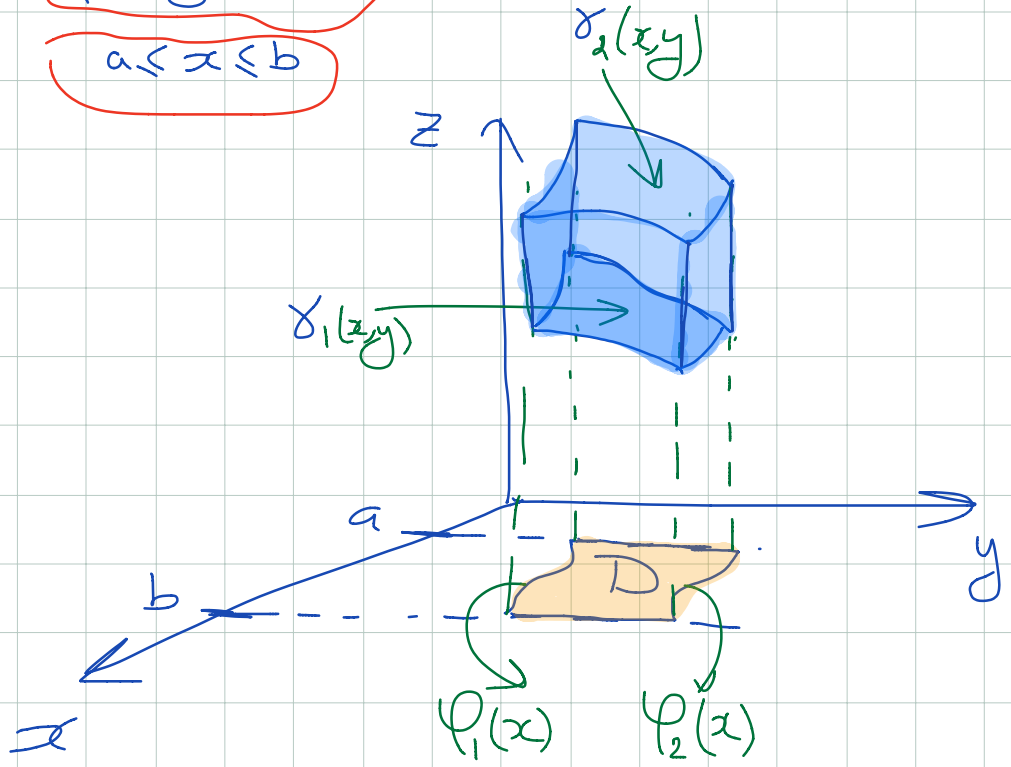
Example: We call a region elementary if it can be described as

$$\gamma_1(x,y) \le z \le \gamma_2(x,y)$$

$(x,y) \in D$ where D is simple.

so $\varphi_1(x) \le y \le \varphi_2(x)$

$a \le x \le b$



Example: The unit ball can be described as an elementary region: bec. $x^2+y^2+z^2 \le 1$ can be written as

$$\begin{cases} -\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2} \\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \\ -1 \le x \le 1 \end{cases} \quad D: \text{y-simple}$$

Physical interpretation of triple integrals:

$$\iiint_W 1 \, dV = \text{Volume}(W)$$

$$\iiint_W f(x,y,z)\, dV = \text{mass of an object } W \text{ with non-homog. density given by } f(x,y,z).$$

Triple Integral by iterated integration

Example: Find the Volume of a ball of radius 1

Sol'n: We want $\iiint_{\text{Ball}} 1 \, dV$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1 \, dz\, dy\, dx$$

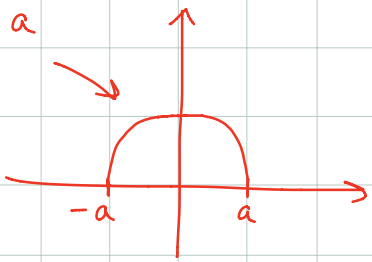
$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2-y^2}\, dy\, dx$$

Let $a = \sqrt{1-x^2}$

$\Rightarrow 2\sqrt{1-x^2-y^2} = 2(a^2-y^2)^{1/2}$

(only to make our calculation easier)

$$= 2\int_{-1}^{1} \underbrace{\int_{-a}^{a} \sqrt{a^2-y^2}\, dy}_{\text{area of semicircle of radius } a}\, dx$$



$$= 2\int_{-1}^{1} \frac{\pi a^2}{2}\, dx \qquad a^2 = 1-x^2$$

$$= \pi \int_{-1}^{1} 1-x^2\, dx = \pi\left(x - x^3/3\right)\Big|_{-1}^{1} = \pi\left(1-\tfrac{1}{3}-\left(-1+\tfrac{1}{3}\right)\right)$$

$$= 4/3\,\pi$$